

Burgers' Equation

Due: Wednesday, February 27, 2019

I. TRAVELING SHOCK IN BURGERS' EQUATION

During our class, we found the steady shock for the 1D Burgers' equation

$$\partial_t u(x, t) + \frac{1}{2} \partial_x u^2 = \nu \partial_x^2 u, \quad (1)$$

when we maintain the velocities $u \rightarrow \pm U$ as $x \rightarrow \mp \infty$.

1) Generalize the solution to the case when we maintain the velocities at infinity $u(-\infty) = U_-$ and $u(+\infty) = U_+$ with $U_- > U_+$.

2) Verify that the dissipation remains finite in the limit $\nu \rightarrow 0$.

3) The velocity of the shock that you have found above is a special case of the Rankine-Hugoniot condition. By integrating the conservation law

$$\partial_t w(x, t) + \partial_x f(w) = 0, \quad (2)$$

around a shock moving with velocity V_s , derive the Rankine-Hugoniot relation $V_s = \frac{f_+ - f_-}{w_+ - w_-}$ where the subscripts \pm refer to the value on the proximal right and left of the shock, respectively.

II. PRE-SHOCK IN BURGERS' EQUATION

During our class, we have shown that the first shock occurs at a time t^* determined by the minimum value of the gradient field at the initial time. The initial gradient and velocity fields are denoted $g_0(a)$ and $u_0(a)$, with $g_0 = \partial_a u_0$. Denote the absolute minimum as $\min_a g_0(a) = -G$ and expand $g_0(a) \simeq -G + \alpha/2a^2 + \dots$ around its minimum, which is assumed to be at the origin.

1) Denote by $X(a, t)$ the position at time t of a particle that was initially at a . Write down the expression for X in terms of a and $u_0(a)$.

2) Re-derive the relation between the first shock time t^* and G .

3) Show that the inverse Lagrangian map $a(X, t)$ at $t = t^*$ behaves singularly around the origin $X = 0$, namely it has a $1/3$ power-law behavior.

4) Show that at the time t^* the velocity $u(X, t)$ also develops a $1/3$ singular behavior at the origin.

5) Show that the enstrophy $\Omega(t) = \int (\partial u / \partial x)^2 dx$ diverges as $(t^* - t)^{-1/2}$ as t approaches t^* .

III. HOPF-COLE TRANSFORMATION FOR BURGERS' EQUATION

Define the stream function as $u = -\partial_x \psi(x, t)$. By taking the space derivative of the Burgers' equation, write down the equation for ψ_t . An unknown function $g(t)$ appears when a space-derivative is factored out. Suppose first that the unknown time-dependent function $g(t) = 0$.

1) Use the Hopf-Cole transformation $\psi(x, t) \equiv 2\nu \log \theta(x, t)$ to reduce the previous equation for ψ_t to the heat equation.

2) Use the Gaussian propagator for the heat equation to obtain the expression of $\theta(x, t)$.

3) In the limit $\nu \rightarrow 0$, show that $\psi(x, t) = \max_a \left[\psi_0(a) - \frac{(x-a)^2}{2t} \right]$. Interpret the max as follows: considers a parabola $\frac{(a-x)^2}{2t} + C$ centered at x , start with a big constant C and reduce it until you contact the curve $\psi_0(a)$ for the first time.

4) Interpret double contacts, i.e. first contact at two different a values for a given x . What is the velocity profile resulting from multiple x 's having the same contact point a ?

5) Can you adapt the arguments in 1) to the case $g(t) \neq 0$?